

# MATH 2060 TUTOR

**Example 1.** Let  $f(x) = \frac{1}{x^3}$  on  $[a, b]$ , where  $0 < a < b$ . Let  $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$  be a partition of  $[a, b]$ . Find tags points  $t_i \in [x_{i-1}, x_i]$  explicitly in terms of  $x_{i-1}, x_i$  such that the corresponding tagged partition  $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$  satisfies

$$S(f; \dot{\mathcal{P}}) = \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right).$$

Hence show that  $f \in \mathcal{R}[a, b]$  and  $\int_a^b f = \int_a^b \frac{1}{x^3} dx = \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right)$ .

**Ans:** Consider the trivial partition  $a < b$  with tag pt.  $c$ .

$$\text{Want: } f(c)(b-a) = \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right)$$

$$\frac{1}{c^3} (b-a) = \frac{b^2 - a^2}{2a^2b^2}$$

$$a^3 = \frac{2a^3b}{2b} < c^3 = \frac{2a^2b^2}{b+a} < \frac{2ab^3}{2a} = b^3$$

$$\text{So } c = \left( \frac{2a^2b^2}{a+b} \right)^{1/3} \in (a, b)$$

This leads us to set

$$t_i := \left( \frac{2x_{i-1}^2 x_i^2}{x_{i-1} + x_i} \right)^{1/3} \text{ for } i=1, \dots, n.$$

Then  $x_{i-1} < t_i < x_i$

So  $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$  satisfies

$$\begin{aligned} S(f; \dot{\mathcal{P}}) &= \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n \frac{1}{2} \left( \frac{1}{x_{i-1}^2} - \frac{1}{x_i^2} \right) \\ &= \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \end{aligned}$$

Now let  $\mathcal{Q} = \{[x_{i-1}, x_i], \xi_i\}_{i=1}^n$  be an arbitrary tagged partition of  $[a, b]$  with  $\|\mathcal{Q}\| < \delta$ .

Define  $\mathcal{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$  as above

Then

$$\begin{aligned} |f(t_i) - f(\xi_i)| &= \left| \frac{1}{t_i^3} - \frac{1}{\xi_i^3} \right| \\ &= |t_i - \xi_i| \frac{|t_i^2 + t_i \xi_i + \xi_i^2|}{t_i^3 \xi_i^3} \\ &< \underline{\delta} \left( \frac{3b^2}{a^6} \right) \quad \text{since } t_i, \xi_i \in [x_{i-1}, x_i]. \end{aligned}$$

Hence

$$\begin{aligned} |S(f; \mathcal{P}) - S(f; \mathcal{Q})| &= \left| \sum_{i=1}^n f(t_i) (x_i - x_{i-1}) - \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}) \right| \\ &\leq \sum_{i=1}^n |f(t_i) - f(\xi_i)| (x_i - x_{i-1}) \\ &< \delta \left( \frac{3b^2}{a^6} \right) \sum_{i=1}^n (x_i - x_{i-1}) \\ &= \delta \left( \frac{3b^2}{a^6} \right) (b-a) \end{aligned}$$

Thus,  $\forall \varepsilon > 0$ , choose  $\delta_\varepsilon := \varepsilon / \left[ \left( \frac{3b^2}{a^6} \right) (b-a) \right]$ ,  
so that if  $\|\mathcal{Q}\| < \delta_\varepsilon$ , then

$$\left| S(f; \mathcal{Q}) - \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \right| < \varepsilon$$

Therefore  $f \in \mathcal{R}[a, b]$  and  $\int_a^b f = \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) =$

### Thm 7.2.1 (Cauchy Criterion)

A fcn  $f : [a, b] \rightarrow \mathbb{R}$  belongs to  $\mathcal{R}[a, b]$  iff

$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0$  s.t.

if  $\mathcal{P}, \mathcal{Q}$  are tagged partitions of  $[a, b]$  with  $\|\mathcal{P}\|, \|\mathcal{Q}\| < \eta_\varepsilon$ ,

then  $|S(f; \mathcal{P}) - S(f; \mathcal{Q})| < \varepsilon$

### Thm 7.2.3 (Squeeze Thm)

Let  $f : [a, b] \rightarrow \mathbb{R}$ . Then  $f \in \mathcal{R}[a, b]$  iff

$\forall \varepsilon > 0, \exists$  fcn  $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a, b]$

with  $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \forall x \in [a, b]$

s.t.  $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

8. Suppose that  $f$  is continuous on  $[a, b]$ , that  $f(x) \geq 0$  for all  $x \in [a, b]$  and that  $\int_a^b f = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ .
9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

8) Ans: Suppose  $f(x_0) > 0 \exists x_0 \in [a, b]$

WLOG, we may assume  $x_0 \in (a, b)$

Then  $\exists \delta > 0$  s.t.

$$f(x) > m := f(x_0)/2 \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a, b].$$

By choosing  $\delta > 0$  small enough, we may assume that  $(x_0 - \delta, x_0 + \delta) \subseteq [a, b]$ .

Consider the step fcn  $\varphi$  on  $[a, b]$  defined by

$$\varphi(x) := \begin{cases} m & , x \in (x_0 - \delta, x_0 + \delta) \\ 0 & , \text{elsewhere} \end{cases}$$

Then  $f, \varphi \in \mathcal{R}[a, b]$  and  $f(x) \geq \varphi(x) \quad \forall x \in [a, b]$ .

By Thm 7.1.5 c)

$$\int_a^b f \geq \int_a^b \varphi$$

$$= m(2\delta) \quad \text{by Lemma 7.2.4}$$

$$> 0,$$

contradicting the assumption that  $\int_a^b f = 0$

Hence  $f(x) = 0 \quad \forall x \in [a, b]$  =

8. Suppose that  $f$  is continuous on  $[a, b]$ , that  $f(x) \geq 0$  for all  $x \in [a, b]$  and that  $\int_a^b f = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ .
9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.

9) Ans: Let  $h: [0, 1] \rightarrow \mathbb{R}$  be given by

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases} \quad \leftarrow \text{not identically zero.}$$

Clearly  $h$  is discontinuous at 0.

Next we show that  $h \in \mathcal{R}[0, 1]$  using Squeeze Thm.

Let  $\varepsilon \in (0, 1)$ .

Let  $\alpha_\varepsilon$  be the constant 0 fcn,  $\alpha_\varepsilon(x) = 0$  on  $[0, 1]$

Let  $w_\varepsilon$  be the step fcn :

$$w_\varepsilon(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

Then  $\alpha_\varepsilon, w_\varepsilon \in \mathcal{R}[0, 1]$  with

$$\alpha_\varepsilon(x) \leq h(x) \leq w_\varepsilon(x) \quad \forall x \in [0, 1].$$

and s.t.

$$\begin{aligned} \int_0^1 (w_\varepsilon - \alpha_\varepsilon) &= \int_0^1 w_\varepsilon \\ &= 1(\varepsilon/2 - 0) = \varepsilon/2 < \varepsilon. \end{aligned}$$

By Squeeze Thm 7.2.3,  $h \in \mathcal{R}[0, 1]$ .

Furthermore, by Thm 7.15

$$0 = \int_0^1 \alpha_\varepsilon \leq \int_0^1 h \leq \int_0^1 w_\varepsilon = \varepsilon/2$$

Letting  $\varepsilon \rightarrow 0^+$ , we have  $\int_0^1 h = 0$ .

## Example

Let a function  $f: [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} \cos^2 x, & x \in [0, \frac{\pi}{2}] \cap \mathbb{Q} \\ 0, & \text{else.} \end{cases}$$

Is this function Riemann integrable?

Ans: Try to apply Cauchy Criterion to show that  $f \notin \mathcal{R}[0, \frac{\pi}{2}]$ .

Let  $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$  be a partition of  $[0, \frac{\pi}{2}]$ ,  
and  $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], g_i\}_{i=1}^n$  be a tagged partition  
with tag pts  $g_i \in [x_{i-1}, x_i] \cap \mathbb{Q}$   
 $\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], r_i\}_{i=1}^n$  be a tagged partition  
with tag pts  $r_i \in [x_{i-1}, x_i] \setminus \mathbb{Q}$

Then

$$\begin{aligned} \int(f; \dot{\mathcal{P}}_1) &= \sum_{i=1}^n f(g_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n \cos^2(g_i)(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^n \frac{1}{2}(x_i - x_{i-1}) \quad \text{assume } x_{k-1} < \frac{\pi}{4} \leq x_k \\ &= \frac{1}{2}(x_{k-1} - x_0) \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{\pi}{4} - x_{k-1}\right) \\ &\geq \frac{\pi}{8} - \frac{1}{2}\|\dot{\mathcal{P}}_1\| \end{aligned}$$

and

$$\int(f; \dot{\mathcal{P}}_2) = \sum_{i=1}^n \cancel{f(r_i)}(x_i - x_{i-1}) = 0$$

(say, uniform partition)

For  $\epsilon_0 = \frac{\pi}{32}$ ,  $\forall \eta > 0$ , we can choose  $\mathcal{P}$  with  $\|\mathcal{P}\| < \min\{\eta, \frac{\pi}{8}\}$

Then  $\exists$  corresponding partitions  $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$  satisfying

$$\|\dot{\mathcal{P}}_1\| = \|\dot{\mathcal{P}}_2\| = \|\mathcal{P}\| < \eta$$

$$\text{s.t. } \left| \int(f; \dot{\mathcal{P}}_1) - \int(f; \dot{\mathcal{P}}_2) \right| \geq \frac{\pi}{8} - \frac{1}{2}\left(\frac{\pi}{8}\right) = \frac{\pi}{16} > \epsilon_0$$

By Cauchy Criterion 7.2.1,  $f \notin \mathcal{R}[0, \frac{\pi}{2}]$

16. If  $f$  is continuous on  $[a, b]$ ,  $a < b$ , show that there exists  $c \in [a, b]$  such that we have  $\int_a^b f = f(c)(b - a)$ . This result is sometimes called the *Mean Value Theorem for Integrals*.

Ans: Note that  $f$  is cts on  $[a, b]$ .

By Extreme Value Thm,  $\exists x_1, x_2 \in [a, b]$  s.t.

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a, b]$$

Note  $f \in R[a, b]$  since it is cts, by Thm 7.2.7.

By Thm 7.15,

$$f(x_1)(b-a) = \int_a^b f(x_1) \leq \int_a^b f \leq \int_a^b f(x_2) = f(x_2)(b-a)$$

$$\Rightarrow f(x_1) \leq \frac{\int_a^b f}{b-a} \leq f(x_2)$$

Now, by Intermediate Value Thm,

$$\exists c \text{ between } x_1, x_2 \Rightarrow c \in [a, b]$$

$$\text{s.t.} \quad f(c) = \frac{\int_a^b f}{b-a}$$

$$\text{i.e.} \quad \int_a^b f = f(c)(b-a) \quad //$$